First-order Linear Differential Equations 一阶线性微分方程（p424）

Standard form (标准形式) of the **first-order linear differential equation**:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

First order: dy/dx

Linear: dy/dx and y occur only to the first power

If $Q(x) = 0$, $\frac{dy}{dx} + P(x)y = 0$ is called homogeneous differential equation

First-order linear homogeneous differential equation(一阶线性齐次微分方程)

$$\frac{dy}{dx} + P(x)y = 0$$

$$\frac{dy}{y} = -P(x)dx \qquad \textit{seperate variables}$$

$$\ln|y| = -\int P(x)dx + C_1$$

$$y = Ce^{-\int P(x)dx} \quad (C = \pm e^{C_1}) \qquad \textit{general solution}$$

Example1: Find the general solution of

$$\frac{dy}{dx} - \frac{2y}{x+1} = 0$$

Solution

$$\frac{dy}{dx} = \frac{2y}{x+1}$$

$$\frac{dy}{y} = \frac{2dx}{x+1}$$

$$\ln|y| = 2\ln|x+1| + C_1$$

$$y = C(x+1)^2 \quad (C = \pm e^{C_1})$$

Variation of Parameters (参数变易法) to solve first-order linear differential equation

In mathematics, variation of parameters, also known as variation of constants, is a general method to solve inhomogeneous linear ordinary differential equations.

The method of variation of parameters was first sketched by the Swiss mathematician Leonhard Euler (1707–1783), and later completed by the Italian-French mathematician Joseph-Louis Lagrange (1736–1813).

$$\frac{dy}{dx}+P(x)y=Q(x) \qquad \textit{inhomogeneous equation}$$

Replace C with $u(x)$ in the general solution of the homogeneous equation:

$$y=u(x)e^{-\int P(x)dx}$$

$$\frac{dy}{dx}=u'e^{-\int P(x)dx}-uP(x)e^{-\int P(x)dx}$$

$$u'e^{-\int P(x)dx}-uP(x)e^{-\int P(x)dx}+P(x)u(x)e^{-\int P(x)dx}=Q(x)$$

$$u'e^{-\int P(x)dx}=Q(x)$$

$$u'=Q(x)e^{\int P(x)dx}$$

$$u=\int Q(x)e^{\int P(x)dx}dx+C$$

Replace C with $u(x)$ in the general solution of the homogeneous equation:

$$y=e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx+C\right)$$

$$y=\underline{Ce^{-\int P(x)dx}}+\underline{e^{-\int P(x)dx}\int Q(x)e^{\int P(x)dx}dx}$$

The General solution of the homogeneous equation

A Particular Solution of the non-homogeneous equation

Example2: Find the general solution of

$$\frac{dy}{dx}-\frac{2y}{x+1}=(x+1)^{5/2}$$

Solution: replace C with $u(x)$ in the general solution of the homogeneous equation

$$y=u(x+1)^2 \qquad -------------- (1)$$

$$\frac{dy}{dx}=u'(x+1)^2+2u(x+1)$$

$$u'(x+1)^2+2u(x+1)-\frac{2u(x+1)^2}{x+1}=(x+1)^{5/2}$$

$$\frac{du}{dx}(x+1)^2=(x+1)^{5/2} \Rightarrow du=(x+1)^{1/2}dx$$

$$u=\frac{x^{3/2}}{3/2}=\frac{2}{3}x^{3/2}+C$$

Plug u in equation (1)

$$y=\left(\frac{2}{3}x^{\frac{3}{2}}+C\right)(x+1)^2 \qquad \text{general soluton}$$

Example3: Find the general solution for the following differential equations

$$\frac{dy}{dx}+y=e^{-x}$$

Step 1: for the homogeneous equation

$$\frac{dy}{dx}+y=0 \ \Rightarrow\ \frac{dy}{dx}=-y \ \Rightarrow\ \frac{dy}{y}=-dx$$

$$\ln|y|=-x+C_1$$

$$y=Ce^{-x}$$

Step 2: for the inhomogeneous equation

$$\frac{dy}{dx}+y=e^{-x}$$

$$y=ue^{-x}$$

$$\frac{dy}{dx}=u'e^{-x}-ue^{-x}$$

Plug in the differential equation

$$u'e^{-x}-ue^{-x}+ue^{-x}=e^{-x} \ \Rightarrow\ \frac{du}{dx}e^{-x}=e^{-x}$$

$$du=dx$$

$$u=x+C$$

$$y=(x+C)e^{-x}$$

$$xy'+y=x^2+3x+2$$

$$y'+y/x=x+3+2/x$$

Step 1: for the homogeneous equation

$$y'+y/x=0$$

$$\frac{dy}{dx}=-\frac{y}{x} \ \Rightarrow\ \frac{dy}{y}=-\frac{dx}{x}$$

$$\ln|y|=-\ln|x|+C_1$$

$$y=C/x$$

Step 2: for the inhomogeneous equation

$$y'+y/x=x+3+2/x$$

$$y=u/x$$

$$\frac{dy}{dx}=\frac{u'}{x}+u\frac{-1}{x^2}$$

Plug in the differential equation

$$\frac{u'}{x}+u\frac{-1}{x^2}+\frac{u}{x^2}=x+3+\frac{2}{x} \ \Rightarrow\ \frac{du}{xdx}=x+3+\frac{2}{x}$$

$$du=(x^2+3x+2)dx$$

$$u=\frac{x^3}{3}+3\frac{x^2}{2}+2x+C$$

$$y=(\frac{x^3}{3}+3\frac{x^2}{2}+2x+C)/x$$

$$y=\frac{x^2}{3}+\frac{3x}{2}+2+\frac{C}{x}$$

$$y'+y\cos x=\mathrm{e}^{-\sin x}$$

Step 1: for the homogeneous equation

$$y'+y\cos x=0$$

$$\frac{dy}{dx}=-y\cos x$$

$$\frac{dy}{y}=-\cos x\,dx$$

$$\ln|y|=-\sin x+C_1$$

$$y=Ce^{-\sin x}$$

Step 2: for the inhomogeneous equation

$$y'+y\cos x=\mathrm{e}^{-\sin x}$$

$$y=ue^{-\sin x}$$

$$dy/dx=u'e^{-\sin x}-ue^{-\sin x}(\cos x)$$

$$u'e^{-\sin x}-ue^{-\sin x}(\cos x)+ue^{-\sin x}\cos x=\mathrm{e}^{-\sin x}$$

$$(du/dx)e^{-\sin x}=\mathrm{e}^{-\sin x} \quad \Rightarrow du=dx$$

$$u=x+C$$

$$y=(x+C)e^{-\sin x}$$

$$y'+y\tan x=\sin 2x$$

$$y=-2\cos^2 x+C\cos x$$

$$(x^2-1)y'+2xy-\cos x=0$$

$$y=(\sin x+C)/(x^2-1)$$

$$dy/dx=\frac{1}{x+y}$$

Solution1: as a first-order linear differential equation

$$\frac{dx}{dy} = x + y \qquad ----------------(1)$$

$$\frac{dx}{dy} - x = y$$ first order linear differential equation

Find the homogeneous solution

$$\frac{dx}{dy} - x = 0$$

$$\frac{dx}{x} = dy$$

$$\ln|x| = y + C_1$$

$$x = e^{y+C_1} = Ce^y$$ general solution of homogeneous equation

Replace C with $u(y)$ in the general solution

$$x = ue^y \qquad -----------------(2)$$

$$\frac{dx}{dy} = u'e^y + ue^y$$

Plug into equation (1)

$$u'e^y + ue^y = ue^y + y$$

$$u'e^y = y \Rightarrow \frac{du}{dy}e^y = y$$

$$du = ye^{-y}dy$$

$$u = \int ye^{-y}dy$$

let $v = e^{-y}, -y = \ln v, dv = -e^{-y}dy$

$$u = \int (-y)(-e^{-y})dy = \int \ln v\, dv = v\ln v - v + C$$

$$u = e^{-y}(-y) - e^{-y} + C$$

Plug into equation (2)

$$x = (-ye^{-y} - e^{-y} + C)e^y = -y - 1 + Ce^y$$

Solution2: change of variables

$$\frac{dy}{dx} = \frac{1}{x+y} \qquad ---------------(1)$$

let $u = x + y, y = u - x, \frac{dy}{dx} = \frac{du}{dx} - 1$

Plug into equation (1)

$$\frac{du}{dx} - 1 = \frac{1}{u} \Rightarrow \frac{du}{dx} = \frac{1}{u} + 1 = \frac{1+u}{u}$$

$$\frac{u}{1+u}du = dx$$

$$x = \int \frac{u}{1+u}du = \int \frac{u+1-1}{1+u}du = \int du - \int \frac{1}{1+u}du$$

$$x = u - \int \frac{1}{1+u}d(1+u) = u - \ln|1+u| + C$$

$$x = x + y - \ln|1 + x + y| + C$$

$$\ln|1 + x + y| = y + C$$

$$1 + x + y = Ce^y$$

$$\frac{dy}{dx} - y\tan x = \sec x\,, \qquad y|_{x=0} = 0$$

$$y = (x + C)\sec x \Rightarrow C = 0$$

$$y = x\sec x$$

$$\frac{dy}{dx} + y\cot x = 5e^{\cos x}\,, \qquad y|_{x=\pi/2} = -4$$

$$y = \frac{(-5e^{\cos x} + C)}{\sin x} \Rightarrow C = -1$$

$$y = \frac{(-5e^{\cos x} - 1)}{\sin x}$$